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A TRIANGULATION OF THE n-CUBE

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This paper is concerned with estimating $\varphi(n)$, the minimum number of n-simplices required to triangulate an n-cube.

1. Introduction

In 1967, H. Scarf developed a finite algorithm for approximating a fixed point of a continuous mapping of a simplex into itself [8]. This algorithm was refined and extended by H. Kuhn, B.C. Eaves and O. Merrill, among others. (See Karamardian [4] and Todd [10] for these and other references.) Several algorithms use a technique of pivoting among simplices which triangulate an n-cube, and there is some hope that reducing the number of simplices in a triangulation may lead to more efficient algorithms. This paper is concerned with estimating $\varphi(n)$, the minimum number of n-simplices required to triangulate an n-cube. We show in particular that

$$L_n \leqslant \varphi(n) \leqslant P_n$$

where $P_1 = 1$ and the P_i satisfy the recursion

$$P_{n+1} = (n+1)(P_n + 2^{n-1}) + 2^{n+1} - (n-1),$$

and where

$$L_n/H_n \approx \tfrac{1}{4}\sqrt{e}\,(n+1)^{\frac{1}{2}},$$

H_n being the number of simplices given by Hadamard's inequality. As is plain from the large gap between them, these bounds are far from sharp. In particular, our upper bound of $0.477(n!)$ should be compared with the fact that $n!$ is commonly quoted [3] as the value $\varphi(n)$. We also show that $\varphi(4) = 16$ and $\varphi(5) \leqslant 67$, thus sharpening the results of Mara [5, 6]. Cottle [1] has also shown that $\varphi(4) = 16$. His proof is interesting in that it uses one of Mara's theorems.

2. The triangulation

We use, without specific reference, a number of basic properties of (convex) polytopes. They can all be found in the book by Grunbaum [2]. In particular, an *n-polytope* is an n-dimensional, compact, convex set with a finite number of extreme points (vertices). And an *n-simplex* is an n-polytope with $n+1$ vertices. An *affine* set is a translate of a linear set. For $A \subseteq \mathbb{R}^n$, aff A (con A) is the intersection of all affine (convex) sets which contain A.

A *triangulation* of an n-polytope P is a finite set S of n-simplices such that
(i) $P = \bigcup S$,
(ii) for all $a, b \in S$, $a \cap b$ is a face of both a and b.

An *n-complex* is a finite set C of n-polytopes such that $P_1 \cap P_2$ is a face of both P_1 and P_2 for all $P_1, P_2 \in C$.

In the rest of this paper, P will be an n-polytope, C an n-complex and S a set of n-simplices unless the context clearly indicates otherwise.

S is a *triangulation* of C if (i) for each $P \in C$ there is a subset of S which triangulates P, and (ii) no proper subset of S has this property. If v is a vertex of P, then a face F of P is *opposite* v if $v \notin F$.

If v is a vertex of P and S is a triangulation of the complex of facets opposite v, then the set $S_v = \{\text{con}(\{v\} \cup s) \colon s \in S\}$ is a triangulation of P.

The notation $A \subset B$ means $A \subseteq B$ and $A \neq B$. If $P = F_n \supset F_{n-1} \supset \cdots \supset F_0 \neq \emptyset$ is a sequence of faces of P and $v_n, v_{n-1}, \ldots, v_0$ is a sequence of vertices of P such that $v_i \in F_i$ and $v_{i+1} \notin F_i$ for $0 \leqslant i \leqslant n-1$, then the v_i's are the vertices of an n-simplex. To see this note that $v_{i+1} \notin \text{aff}\{v_i, \ldots, v_0\}$ for $0 \leqslant i \leqslant n-1$.

These ideas lead naturally to the following construction of a triangulation for a complex C. Let $v_1, v_2, \ldots, v_L$ be an ordering of the set of vertices of the polytopes of C. For each face $F \neq \emptyset$ of $P \in C$, let $i_F = \min\{k \colon v_k \in F\}$ and $V_F = v_{i_F}$. Each sequence of faces $P = F_n \supset \cdots \supset F_0 \neq \emptyset$ with $v_{F_{i+1}} \notin F_i$ for $0 \leqslant i \leqslant n-1$ has an associated simplex $s = \text{con}(\{v_{F_n}, \ldots, v_{F_0}\})$. Let $S_C(v_1, \ldots, v_L)$ be the set of all simplices generated by sequences of faces as defined above.

Lemma. *For any complex C, the set $S_C(v_1, \ldots, v_L)$ is a triangulation of C.*

Proof. By induction. For $n = 2$, the lemma is obviously true.

Assume that the result holds for n and let C be a complex of $(n+1)$-polytopes. Let C_n be the complex of facets of the polytopes of C. Then $S_{C_n}(v_1, \ldots, v_L)$ is a triangulation of C_n. For any polytope $P \in C$, the complex of facets opposite v_P is triangulated by a subset S' of $S_{C_n}(v_1, \ldots, v_L)$. Each simplex in S' has an associated sequence of faces $F_n \supset \cdots \supset F_0 \neq \emptyset$. Since $v_P \notin F_n$, this sequence can be extended to the sequence $P = F_{n+1} \supset F_n \supset \cdots \supset F_0$. Thus

$$S_P = \{\text{con}(\{v_P\} \cup s) \colon s \in S'\} \subseteq S_C(v_1, \ldots, v_L)$$

is a triangulation of P.

In order to guarantee that $S_C(v_1, \ldots, v_L)$ is a triangulation of C, we must still show that if $s = \mathrm{con}(\{v_P, v_{F_{n-1}}, \ldots, v_{F_0}\})$ is a simplex in $S_{c_r}(v_1, \ldots, v_L)$ containing v_P, then there is an associated sequence of faces $P = F_{n+1} \supset F'_n \supset \cdots \supset F'_0$ such that $\{v_P, v_{F_{n-1}}, \ldots, v_{F_0}\} \subset \{v_P, v_{F'_n}, \ldots, v_{F'_0}\}$.

Clearly $\{v_{F_{n-1}}, \ldots, v_{F_0}\}$ is a set of vertices in an $(n-1)$-face of P which does not contain v_P. Thus there is an n-face F'_n of P which contains this set and not v_P. If $V_{F_n} = v_{F_{n-1}}$, then $P \supset F'_n \supset F_{n-1} \supset \cdots \supset F_0$ is the required sequence of faces. If $v_{F'_n} = v_{F_{n-1}}$, then by induction there is a sequence of faces of F'_n such that $F'_n \supset \cdots \supset F'_0$ with $\{v_{F_{n-1}}, \ldots, v_{F_0}\} \subset \{v_{F'_n}, \ldots, v_{F'_0}\}$. Hence $\{v_p, v_{F_{n-1}}, \ldots, v_{F_0}\} \subset \{v_p, v_{F'_n}, \ldots, v_{F'_0}\}$. This proves the lemma.

Theorem. *Let $\varphi(n)$ be the minimum number of simplices required to triangulate an n-cube. Then $\varphi(n) \leq P_n$, where $P_1 = 1$ and*

$$P_{n+1} = (n+1)(P_n - 2^{n-1}) + 2^{n+1} - (n+1) \quad \textit{for } n \geq 1.$$

Proof. For the n-cube $I^n = [0, 1]^n$, we will call a vertex $v = (v_1, \ldots, v_n)$ odd if an odd number of the v_i's are 1 and even otherwise. For each vertex v of I^n there is a hyperplane which passes through the vertices adjacent to it. For example, the hyperplane $H = \{x: x_1 + \cdots + x_n = 1\}$ passes through the vertices adjacent to $0 = (0, \ldots, 0)$. It can be shown by a simple induction that $H \cap I^n$ is the $(n-1)$-simplex whose vertices are the vertices of I^n adjacent to 0. For $n \geq 3$, let $\mathcal{O}$ be the set of odd vertices of I^n and for each $v \in \mathcal{O}$, let H_v be the hyperplane which passes through the vertices adjacent to v. Let H_v^- be the closed halfspace which contains v and H_v^+ be the closed halfspace which contains the rest of I^n. Let T^n denote the truncated cube $I^n \bigcap_{v \in \mathcal{O}} H_v^+$. It can be shown that $T^n = \mathrm{con}(E)$, where E is the set of even vertices of I^n. If $A^n = \{I^n \cap H_v^-: v \in \mathcal{O}\}$, then A^n is a set of simplices and $|A^n| = 2^{n-1}$. Finally, let $C^n = \{T^n\} \cup A^n$. It is clear that $I^n = \bigcup C^n$ and that a triangulation of C^n is a triangulation of I^n.

Let S^n be the triangulation of C^n given by the lemma and $P_n = |S^n|$. Clearly, $A^n \subseteq S^n$, thus $S^n - A^n$ is a triangulation of T^n and $|S^n - A^n| = P_n - 2^{n-1}$. Consider the triangulation S^{n+1} of C^{n+1}. To triangulate T^{n+1} we note that T^{n+1} has 2^n simplex faces and $2(n+1)$ faces congruent to T^n. Of these faces, $n+1$ simplex faces and $n+1$ T^n-faces are adjacent to $v_{T^{n+1}}$. Each T^n-face opposite $V_{T^{n+1}}$ will generate $P_n - 2^{n-1}$ simplices and the $2^n - (n+1)$ simplex faces opposite v_{n+1} will each generate one simplex for a total of $(n+1)(P_n - 2^{n-1}) + 2^n - (n+1)$ simplices in $S^{n+1} - A^{n+1}$ which triangulate T^{n+1}. Thus

$$\begin{aligned} P_{n+1} &= |S^{n+1}| = |S^{n+1} - A^{n+1}| + |A^{n+1}| \\ &= (n+1)(P_n - 2^{n-1}) + 2^n - (n+1) + 2^n \\ &= (n+1)(P_n - 2^{n-1}) + 2^{n+1} - (n+1). \end{aligned}$$

3. An upper bound

The common triangulations of I^n require $n!$ simplices. The first ten values of P_n are 1, 2, 5, 16, 67, 364, 2445, 19296, 173015 and 1720924.

Consider the ratio $P_n/n!$. Since

$$\frac{P_{n+1}}{(n+1)!}=\frac{P_n}{n!}-\frac{2^{n-1}}{n!}+\frac{2^{n+1}}{(n+1)!}-\frac{1}{n!}\leqslant\frac{P_n}{n!}$$

for $n \geqslant 3$, the sequence is decreasing. We also have

$$\frac{P_{n+1}}{(n+1)!}\geqslant\frac{P_n}{n!}-\frac{2^{n-1}}{n!}-\frac{1}{n!}\geqslant\frac{P_n}{n!}-\frac{2^n}{n!}$$

and by induction

$$\frac{P_{n+k}}{(n+k)!}\geqslant\frac{P_n}{n!}-\frac{2^{n+i}}{(n+i)!}\geqslant\frac{P_n}{n!}-\sum_{i=n}^{\infty}\frac{2^i}{i!}$$
$$\geqslant\frac{P_n}{n!}-\left(e^2-\sum_{i=0}^{n-1}\frac{2^i}{i!}\right).$$

For $n=10$,

$$\frac{P_n}{n!}=0.47636 \quad\text{and}\quad e^2-\sum_{i=1}^{9}\frac{2^i}{i!}=0.00034.$$

Thus $P_n/n! \in [0.47600, 0.47636]$ for $n \geqslant 10$.

4. A lower bound

From Hadamard's inequality, if M is a $(1,-1)$ $n\times n$ matrix, then $|\det(M)|\leqslant n^{n/2}$. Thus any $(0,1)$ $n\times n$ matrix M satisfies the inequality $|\det(m)|\leqslant n^{n/2}/2^{n-1}$. By the well-known formula [9] for the volume of a simplex, the maximum volume, V_n, of a simplex in I^n satisfies $V_n\leqslant(1/n!)\,|\det(M)|$, where M is a $(0,1)$ $(n+1)\times(n+1)$ matrix. Let

$$H_n=(2^n\cdot n!)/(n+1)^{(n+1)/2}.$$

Then $\varphi(n)\geqslant 1/V_n\geqslant H_n$ and H_n/P_n approaches 0 as n becomes large, which shows that there is a huge gap between H_n and P_n.

However, if we consider only triangulations of I^n whose vertices coincide with those of I^n we can do slightly better. For $K\subseteq R^n$, let $V(K)$ denote the volume of K. If K is a set of subsets of R^n, we will use $V(K)$ instead of $V(\cup K)$. If F is a facet of the polytope P and S is a triangulation of P, we will say that a simplex $s\in S$ *belongs* to F if a facet of s is a subset of F. If S_1 and S_2 are the sets of simplices belonging to two adjacent facets of I^n, then

$$V(S_1)=1/n \quad\text{and}\quad V(S_1\cap S_2)\leqslant 1/n(n-1).$$

Theorem. *If S is a triangulation of I^n whose vertices coincide with those of I^n and $L_n=\frac{1}{2}(H_n+n\cdot H_{n-1})$, then $|S|\geqslant L_n$ and $L_n/H_n\approx\frac{1}{2}+\frac{1}{2}\sqrt{e}(n+1)^{\frac{1}{2}}$ for large n.*

Proof. Let F_{i_1} and F_{i_2} be distinct pairs of opposite facets of I^n for $1\leqslant i\leqslant n$ and

$A_i = \{s \in S$: s belongs to F_{i_1} or $F_{i_2}\}$. Then for each i, $V(A_i) = 2/n$. If $B_k = \bigcup_{i=1}^{k} A_i$, then

$$V(B_k) \geq \sum_{i=1}^{k} \left(\frac{2}{n} - \frac{4(i-1)}{n(n-1)} \right).$$

To see this last inequality, note that $B_k = B_{k-1} \cup A_k$. There are $4(k-1)$ intersections of sets belonging to the two facets which determine A_k and the $2(k-1)$ facets which determine B_{k-1}. Hence

$$\begin{aligned} V(B_k) &= V(B_{k-1}) + V(A_k) - V(B_{k-1} \cap A_k) \\ &\geq \sum_{i=1}^{k-1} \left(\frac{2}{n} - \frac{4(i-1)}{n(n-1)} \right) + \left(\frac{2}{n} - \frac{4(k-1)}{n(n-1)} \right) \end{aligned}$$

It is a straightforward exercise to show that $V(B_k) \geq \frac{1}{2}$ for $k = \lfloor \frac{1}{2}(n+1) \rfloor$. Thus the total volumes of the simplices in S which belong to at least one facet must be at least $\frac{1}{2}$. The volume of a simplex which belongs to a facet is less than or equal to $(1/n)V_{n-1}$, thus $|S| \geq \frac{1}{2}(H_n + nH_{n-1})$ and the theorem follows.

Finding a lower bound for $\varphi(n)$ significantly better than H_n seems to be a difficult problem.

5. Dimension $n \leq 6$

For $n = 3$ and $n = 4$, it can be shown that the minimum triangulations require 5 and 16 simplices respectively, thus

$$\varphi(3) = P_3 \quad \text{and} \quad \varphi(4) = P_4.$$

For $n = 3$, $|B_1| \geq 4$ and $V(B_1) \leq \frac{2}{3}$, showing that there must be at least one more simplex to fill I^n. For $n = 4$, we use the facts that the maximum volume of a simplex is $\frac{3}{24}$ and that if a facet has exactly five simplices, the simplex with base having area $\frac{1}{3}$ can belong to only that facet. The fact that $\varphi(4) \geq 16$ then follows from a straightforward consideration of the following cases: all facets have 5 simplices; there is a facet having 6 or more simplices, but no opposite pair of facets each having 6 or more simplices; there is a pair of opposite facets with 6 or more simplices each. For example, in the second case, there must be at least 11 simplices with total volume less than or equal to $\frac{1}{2}$. Since no two opposite facets have 6 or more simplices, there must be at least 4 simplices with bases of area 2, one has been counted and the other 3 have total volume less than or equal to $\frac{1}{4}$ since the maximum altitude of a simplex belonging to a facet is 1. It is clear that there must be at least 2 more simplices to fill the remainder of I^4.

For $n = 5$, $\varphi(5) \leq P_5 = 67$. It is not known whether or not equality holds.

For $n = 6$, since there is a triangulation of I^6 with only 344 simplices while $P_6 = 364$, it is likely that $\varphi(n) < P_n$ for all $n \geq 6$.

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